Partially Balanced Incomplete Block Design (PBIBD)

Why PBIBD?

- Sometimes, BIBDs require large number of replications.

v=8, k=3

$$b = \binom{8}{3} = 56$$

$$r = \frac{bk}{v} = 21.$$

Partially balanced Association Schemes(m-associate classes)

- Any two symbols are either first, second,…, or m^{th} associates & the relation of association is symmetrical.
- Each treatment has exactly n_i treatments, n_i does not depend on the treatments.
- p^i_{jk} is independent of the pair of i^{th} associates.

The numbers $v, n_1, n_2, \ldots, n_m, p^i_{jk}$ $(i,j,k=1,2,\ldots m)$

- parameters of m-associate partially balanced scheme.

Rectangular Association Scheme

m=3

1		2		3
4		5		6

Under this arrangement, with respect to each symbol,

- first associates: same row
- second associates: same column
- third associates: remaining symbols

1		2		3
4		5		6

The table below describes the first, second and third associates of all the six treatments.

Treatment number	First associates	Second associates	Third associates
1	2, 3	4	5, 6
2	1, 3	5	4, 6
3	1, 2	6	4, 5
4	5, 6	1	2, 3
5	4, 6	2	1, 3
6	4, 5	3	1, 2

- $n_1=2$, $n_2=1$ & $n_3=2$
- holds true for other treatments too.

The table below describes the first, second and third associates of all the six treatments.

Treatment number	First associates	Second associates	Third associates
1	2, 3	4	5, 6
2	1, 3	5	4, 6
3	1, 2	6	4, 5
4	5, 6	1	2, 3
5	4, 6	2	1, 3
6	4, 5	3	1, 2

- (condition (iii) of definition of partially balanced association schemes) p^i_{jk} is independent of the pair of i^{th} associates.

- p^1_{33} = 1. (take egs, treatments 1 &2; treatments 2&3)

Triangular Association Scheme

- 2 – class association scheme

- v treatments arranged in q rows and q columns where:

$$v = \binom{q}{2} = \frac{q(q-1)}{2}.$$

These symbols are arranged as follows:

- leading diagonals left blank .
- (q/2) positions filled above the principal diagonal by the treatment numbers 1,2,…,v.
- Rest filled symmetrically.

Rows → / Columns ↓	1	2	3	4	...	$q - 1$	q
1	×	1	2	3	...	$q - 2$	$q - 1$
2	1	×	q	$q + 1$	...	$2q - 2$	$2q - 1$
3	2	q	×	...	...	...	...
4	3	$q + 1$	...	...	...	...	...
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
$q - 1$	$q - 2$	$2q - 2$	...	...	...	×	$q(q - 1)/2$
q	$q - 1$	$2q - 1$	...	...	...	$q(q-1)/2$	×

first associates : same column
second associates: remaining

10 treatments

$$q = 5 \text{ as } v = \binom{5}{2} = 10.$$

Rows → / Columns ↓	1	2	3	4	5
1	×	1	2	3	4
2	1	×	5	6	7
3	2	5	×	8	9
4	3	6	8	×	10
5	4	7	9	10	×

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5

Six parameters p^1_{11}, p^1_{22}, p^1_{12}(or p^1_{21}),p^2_{11},p^2_{22} and p^2_{12}(or p^2_{21}) arranged in symmetric matrices P_1 & P_2 as

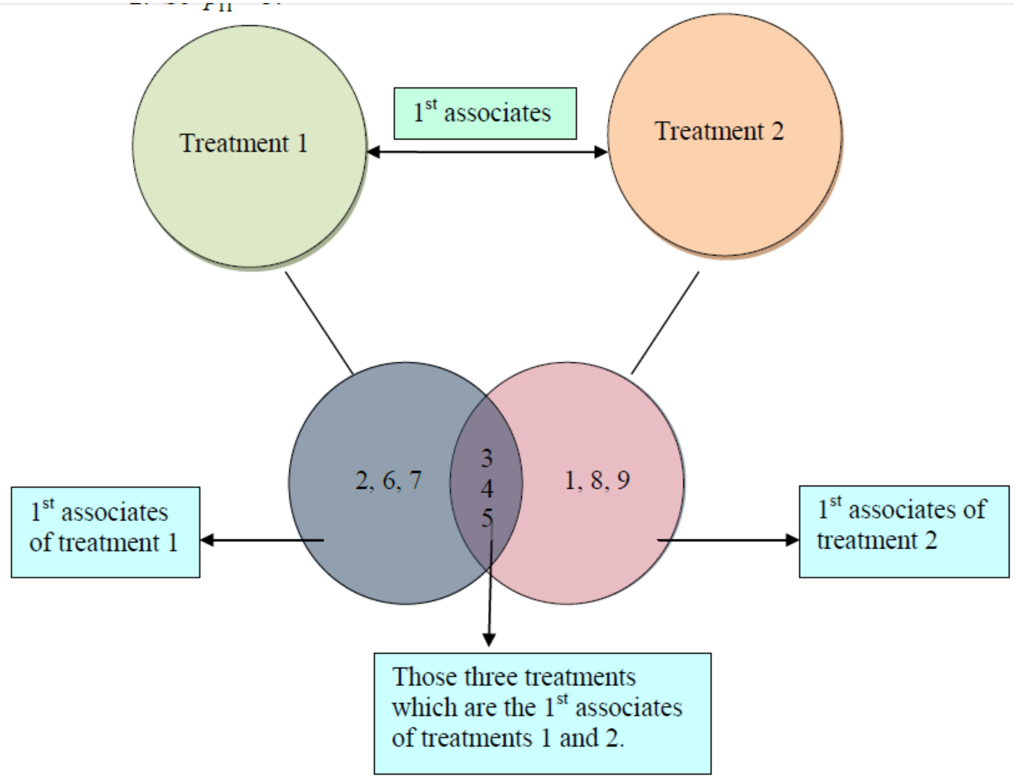
$$P_1 = \begin{bmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{bmatrix}, P_2 = \begin{bmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{bmatrix}$$

Here,

$$P_1 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}$$

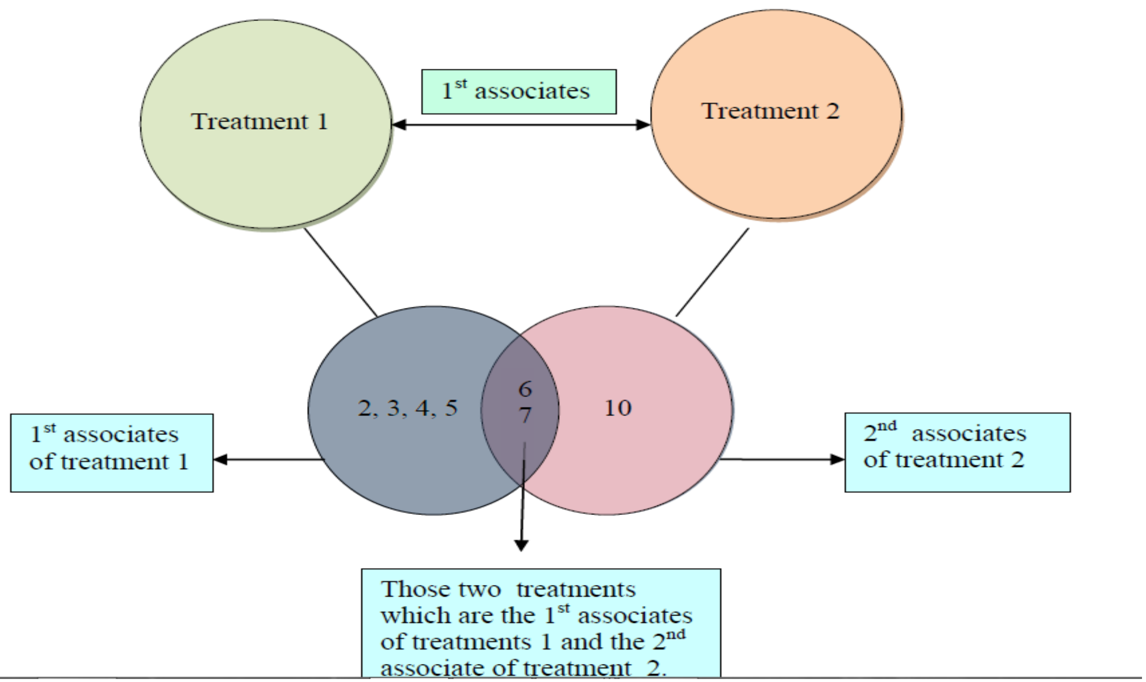
p^1_{11}

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5



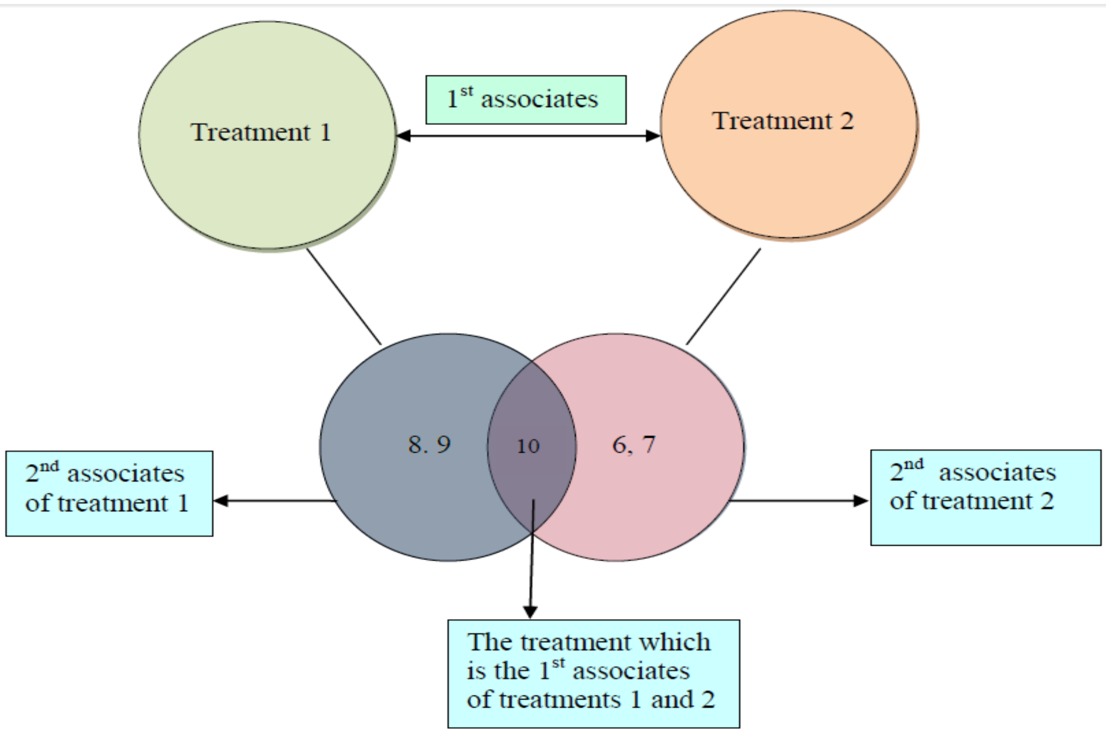
p^1_{12} & p^1_{21}

Treatment number	First associates		Second associates
1	2, 3, 4	5, 6, 7	8, 9, 10
2	1, 3, 4	5, 8, 9	6, 7, 10
3	1, 2, 4	6, 8, 10	5, 7, 9
4	1, 2, 3	7, 9, 10	5, 6, 8
5	1, 6, 7	2, 8, 9	3, 4, 10
6	1, 5, 7	3, 8, 10	2, 4, 9
7	1, 5, 6	4, 9, 10	2, 3, 8
8	2, 5, 9	3, 6, 10	1, 4, 7
9	2, 5, 8	4, 7, 10	1, 3, 6
10	3, 6, 8	4, 7, 9	1, 2, 5



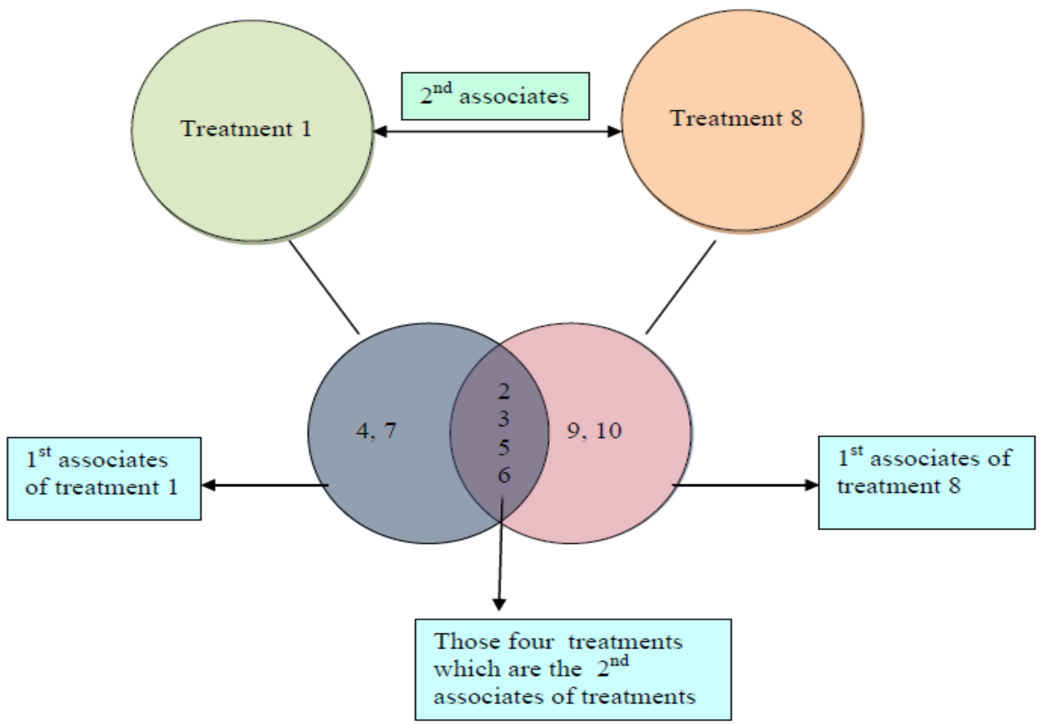
p^1_{22}

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5



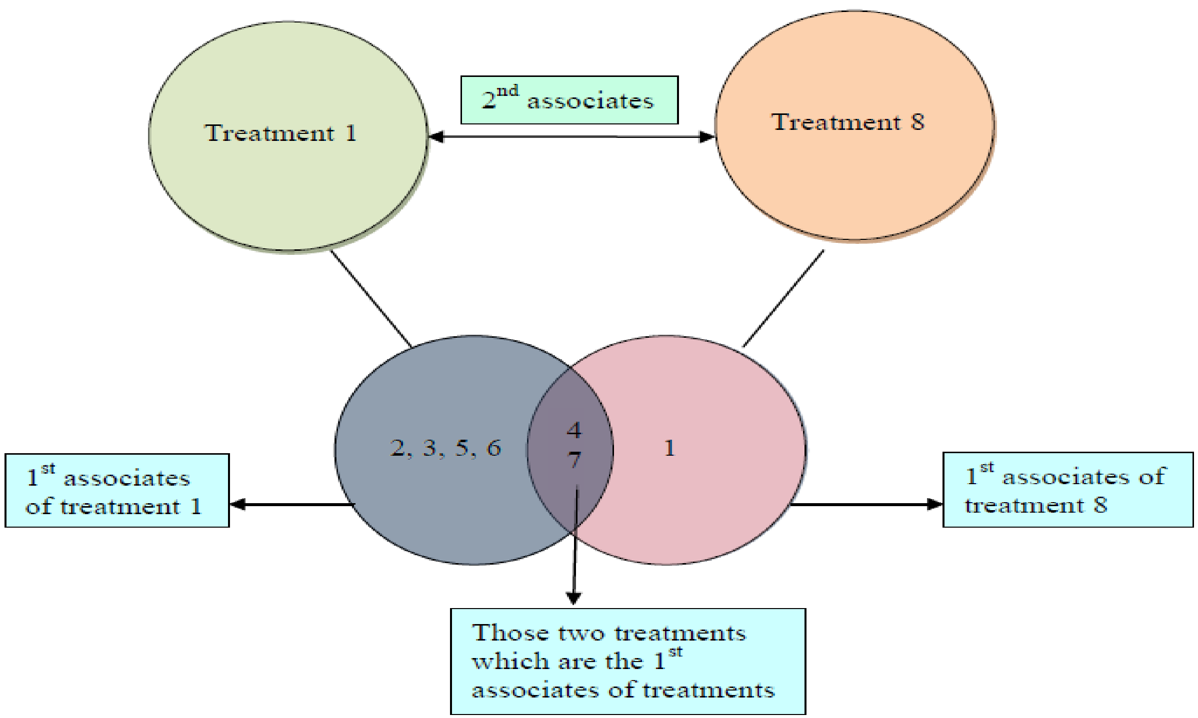
p^2_{11}

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5



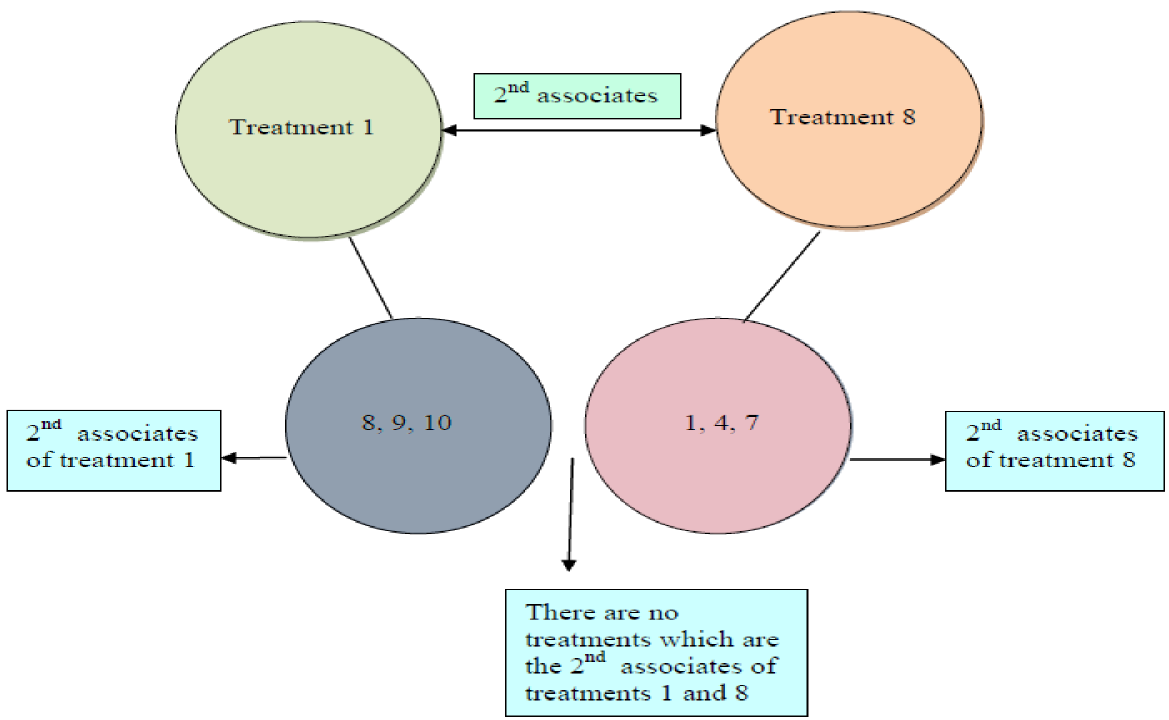
p^2_{12} & p^2_{21}

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5



p^2_{22}

Treatment number	First associates	Second associates
1	2, 3, 4 5, 6, 7	8, 9, 10
2	1, 3, 4 5, 8, 9	6, 7, 10
3	1, 2, 4 6, 8, 10	5, 7, 9
4	1, 2, 3 7, 9, 10	5, 6, 8
5	1, 6, 7 2, 8, 9	3, 4, 10
6	1, 5, 7 3, 8, 10	2, 4, 9
7	1, 5, 6 4, 9, 10	2, 3, 8
8	2, 5, 9 3, 6, 10	1, 4, 7
9	2, 5, 8 4, 7, 10	1, 3, 6
10	3, 6, 8 4, 7, 9	1, 2, 5



Rows $\rightarrow$ Columns $\downarrow$	1	2	3	4	$\cdots$	$q-1$	q
1	$\times$	1	2	3	$\cdots$	$q-2$	$q-1$
2	1	$\times$	q	$q+1$	$\cdots$	$2q-2$	$2q-1$
3	2	q	$\times$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
4	3	$q+1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$q-1$	$q-2$	$2q-2$	$\cdots$	$\cdots$	$\cdots$	$\times$	$q(q-1)/2$
q	$q-1$	$2q-1$	$\cdots$	$\cdots$	$\cdots$	$q(q-1)/2$	$\times$

In general, if q rows and q columns of a square are used, then for $q > 3$

$$v = \binom{q}{2} = \frac{q(q-1)}{2},$$

$$n_1 = 2q-4,$$

$$n_2 = \frac{(q-2)(q-3)}{2},$$

$$P_1 = \begin{bmatrix} q-2 & q-3 \\ q-3 & \dfrac{(q-3)(q-4)}{2} \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 4 & 2q-8 \\ 2q-8 & \dfrac{(q-4)(q-5)}{2} \end{bmatrix}.$$

Construction of Blocks of PBIBD under Triangular Association Scheme

- Approach 1

Rows → / Columns ↓	1	2	3	4	5
1	×	1	2	3	4
2	1	×	5	6	7
3	2	5	×	8	9
4	3	6	8	×	10
5	4	7	9	10	×

Blocks	Treatments
Block 1	1, 2, 3, 4
Block 2	1, 5, 6, 7
Block 3	2, 5, 8, 9
Block 4	3, 6, 8, 10
Block 5	4, 7, 9, 10

Approach 2

Rows → Columns ↓	1	2	3	4	5
1	×	1	2	3	4
2	1	×	5	6	7
3	2	5	×	8	9
4	3	6	8	×	10
5	4	7	9	10	×

Blocks	Columns of association scheme	Treatments
Block 1	(1, 2)	2, 3, 4, 5, 6, 7
Block 2	(1, 3)	1, 3, 4, 5, 8, 9
Block 3	(1, 4)	1, 2, 4, 6, 8, 10
Block 4	(1, 5)	1, 2, 3, 7, 9, 10
Block 5	(2, 3)	1, 2, 6, 7, 8, 9
Block 6	(2, 4)	1, 3, 5, 7, 8, 10
Block 7	(2, 5)	1, 4, 5, 6, 9, 10
Block 8	(3, 4)	2, 3, 5, 6, 9, 10
Block 9	(3, 5)	2, 4, 5, 7, 8, 10
Block 10	(4, 5)	3, 4, 6, 7, 8, 9

Approach 3 (consider all 1st associates of given treatment)

Rows → / Columns ↓	1	2	3	4	5
1	×	1	2	3	4
2	1	×	5	6	7
3	2	5	×	8	9
4	3	6	8	×	10
5	4	7	9	10	×

Blocks	Columns of association scheme	Treatments
Block 1	(1, 2)	2, 3, 4, 5, 6, 7
Block 2	(1, 3)	1, 3, 4, 5, 8, 9
Block 3	(1, 4)	1, 2, 4, 6, 8, 10
Block 4	(1, 5)	1, 2, 3, 7, 9, 10
Block 5	(2, 3)	1, 2, 6, 7, 8, 9
Block 6	(2, 4)	1, 3, 5, 7, 8, 10
Block 7	(2, 5)	1, 4, 5, 6, 9, 10
Block 8	(3, 4)	2, 3, 5, 6, 9, 10
Block 9	(3, 5)	2, 4, 5, 7, 8, 10
Block 10	(4, 5)	3, 4, 6, 7, 8, 9

PBIBD with m-associate classes

- v,b,k
- v treatments arranged in b blocks according to m-associate partially balanced association

a) Every treatment occurs at most once in a block

b) Every treatment occurs exactly in r blocks

c) If two treatments are the i^{th} associates, they occur in exactly λ_i (i=1,2,…,m) blocks.

- $b, v, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m, n_1, n_2, \ldots, n_m$: parameters of first kind
  p^i_{jk} : parameters of second kind

- $\lambda_i = \lambda$ for all $i=1,2,\ldots,m$ then PBIBD reduces to BIBD.

Conditions for PBIBD

$$(i) \quad bk = vr$$

$$(ii) \quad \sum_{i=1}^{m} n_i = v-1$$

$$(iii) \quad \sum_{i=1}^{m} n_i \lambda_i = r(k-1)$$

$$(iv) \quad n_k p^k_{ij} = n_i p^i_{jk} = n_j p^j_{ki}$$

$$(v) \quad \sum_{k=1}^{m} p^i_{jk} = \begin{cases} n_j - 1 & \text{if } i = j \\ \mathrm{n_j} & \text{if } i \neq j. \end{cases}$$

Interpretation of Conditions for PBIBD

(i) $bk = vr$ (same as BIBD)

(ii) $\sum_{i=1}^{m} n_i = v - 1$ (w.r.t to each treatment, the remaining (v-1) treatments classified as first, second,…,m^{th} associates; each treatment has n_i associates)

(iii) $\sum_{i=1}^{m} n_i \lambda_i = r(k-1)$ (A occurs in r blocks, r blocks contain r(k-1) pairs of treatments; the n_i associates of A occurs in λ_i times)